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tt^*-geometry on the tangent bundle of an almost complex manifold

Lars Schäfer*

Mathematisches Institut der Universität Bonn, Beringstraße 1, D-53115 Bonn, Germany
Institut Élie Cartan de Mathématiques, Université Henri Poincaré - Nancy 1, B.P. 239, F-54506 Vandœuvre-lès-Nancy Cedex, France

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Abstract

The subject of this paper is tt^*-bundles (TM, D, S) over an almost complex manifold (M, J). Let ∇ be a flat connection on M. We characterize those tt^*-bundles with $\nabla = D + S$ which are induced by the one parameter family of connections $\nabla^\theta = \exp(\theta J) \circ \nabla \circ \exp(-\theta J)$ and obtain a uniqueness result for solutions where D is complex. A subclass of such solutions is flat nearly Kähler manifolds and special Kähler manifolds. Moreover, we study the case where these tt^*-bundles admit the structure of symplectic or metric tt^*-bundles. Finally, we generalize the notion of pluriharmonic maps to maps from almost complex manifolds (M, J) into pseudo-Riemannian manifolds and relate the above symplectic and metric tt^*-bundles to pluriharmonic maps from (M, J) into the pseudo-Riemannian symmetric spaces $SO_0(p, q)/U(p, q)$ and $\mathrm{Sp}(\mathbb{R}^{2n})/U(p, q)$, respectively.

Keywords: tt^*-geometry and tt^*-bundles; Special complex and special Kähler manifolds; Nearly Kähler manifolds; Pluriharmonic maps; Pseudo-Riemannian symmetric spaces

1. Introduction

In this work we study *tt^*-bundles* on the tangent bundle of an almost complex manifold (M, J) as base. In previous work about tt^*-bundles we only considered the case where (M, J) is a complex manifold [4,17]. However, in the study of tt^*-bundles on the tangent bundle it is reasonable to consider almost complex manifolds, since in this way *nearly Kähler manifolds* with flat Levi-Civita connection arise as solutions of tt^*-geometry. We give a constructive classification of Levi-Civita flat nearly Kähler manifolds in a common work with Cortés [5]. Another class of interesting solutions is *special Kähler* manifolds which we studied in [4]. In other words, tt^*-bundles on the tangent bundle of an almost complex manifold (M, J) are a common generalization of these two important geometries. Nearly Kähler manifolds are of interest in the physics of string theory and supersymmetry (compare Friedrich and Ivanov [10] and references within) and in mathematics of weak holonomy (compare the works of A. Gray). The notion of special Kähler manifolds was introduced by de Wit and van Proeyen [7] and has its origin in certain supersymmetric field theories. For a survey on this subject we refer to Cortés [3].

* Corresponding address: Department Mathematik, Universität Hamburg, Bundesstraße 55, D-20146 Hamburg, Germany.
E-mail addresses: schaefer@math.uni-bonn.de, schafer@iecn.u-nancy.fr.

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Let us explain the structure of this paper. First we introduce the notion of (metric, symplectic) tt^*-bundles and give explicit equations for this geometric structure, the so-called tt^*-equations. Part of the tt^*-data is a one-parameter family of flat connections D^θ with $\theta \in \mathbb{R}$. Every almost complex manifold (M, J) endowed with a flat connection ∇ carries a natural family of flat connections given by

$$\nabla^\theta = \exp(\theta J) \circ \nabla \circ \exp(-\theta J), \quad \text{with } \theta \in \mathbb{R}.$$

We study tt^*-bundles for which the families D^θ and ∇^θ are equivalent in the sense of the following:

Definition 1. Two one-parameter families of connections ∇^θ, D^θ on some vector bundle E with $\theta \in \mathbb{R}$ are called (linear) equivalent with factor $\alpha \in \mathbb{R}$ if they satisfy the equation $\nabla^\theta = D^{\alpha\theta}$.

Afterwards we restrict to tt^*-bundles as above such that the connection $D := \frac{1}{2}(D^0 + D^\pi)$ is complex, i.e. satisfies $DJ = 0$. These are recovered uniquely from the connection ∇ and the complex structure J. In addition compatibility conditions on the pair (∇, J) are given and it is shown that for *special complex* and nearly Kähler manifolds these compatibility conditions on (∇, J) hold. More precisely, we give a class of solutions which corresponds to special complex manifolds with torsion and a class of solutions which corresponds to flat almost complex manifolds satisfying the nearly Kähler condition (with torsion). In the following we study whether these tt^*-bundles provide *metric* and *symplectic* tt^*-bundles. Solutions of the first type are, for example, given by special Kähler manifolds and the second arise on Levi-Civita flat nearly Kähler manifolds, while neither special Kähler manifolds admit symplectic tt^*-bundles nor Levi-Civita flat nearly Kähler manifolds admit metric tt^*-bundles.

Finally, there is a relation between pluriharmonic maps and tt^*-geometry, which was studied in [6,17,18,4]. In this work we generalize the notion of a *pluriharmonic map* to maps from almost complex manifolds into pseudo-Riemannian manifolds. We introduce $\mathbb{S}^1$-pluriharmonic maps which generalize the notion of associated families of pluriharmonic maps (see for example [8]) to maps from almost complex manifolds into pseudo-Riemannian manifolds. We give conditions for an $\mathbb{S}^1$-pluriharmonic map to be pluriharmonic and a result, which relates generalized pluriharmonic to harmonic maps. With these notions we associate pluriharmonic maps into $\mathrm{Sp}(\mathbb{R}^{2n})/U(p,q)$, respectively $SO_0(p,q)/U(p,q)$, to the above metric and symplectic tt^*-bundles.

2. tt^*-bundles

We extend the real differential–geometric notion of a tt^*-bundle of [4,17] by admitting the complex structure of the base manifold to be non-integrable. Further we introduce the notion of symplectic tt^*-bundles.

Definition 2. A tt*-bundle (E, D, S) over an almost complex manifold (M, J) is a real vector bundle $E \to M$ endowed with a connection D and a section $S \in \Gamma(T^*M \otimes \mathrm{End}\, E)$ which satisfy the tt*-equation

$$R^\theta = 0 \quad \text{for all } \theta \in \mathbb{R}, \tag{2.1}$$

where R^θ is the curvature tensor of the connection D^θ defined by

$$D^\theta_X := D_X + \cos(\theta) S_X + \sin(\theta) S_{JX} \quad \text{for all } X \in TM. \tag{2.2}$$

A metric tt*-bundle (E, D, S, g) is a tt*-bundle (E, D, S) endowed with a possibly indefinite D-parallel fiber metric g such that for all $p \in M$

$$g(S_X Y, Z) = g(Y, S_X Z) \quad \text{for all } X, Y, Z \in T_p M. \tag{2.3}$$

A symplectic tt*-bundle (E, D, S, ω) is a tt*-bundle (E, D, S) endowed with the structure of a symplectic vector bundle[1] (E, ω), such that ω is D-parallel and S is ω-symmetric, i.e. for all $p \in M$

$$\omega(S_X Y, Z) = \omega(Y, S_X Z) \quad \text{for all } X, Y, Z \in T_p M. \tag{2.4}$$

[1] see Mc Duff and Salamon [14].

Remark 1. If (E, D, S) is a tt*-bundle then (E, D, S^θ) is a tt*-bundle for all $\theta \in \mathbb{R}$, where

$$S^\theta := D^\theta - D = \cos(\theta) S + \sin(\theta) S_J. \tag{2.5}$$

The same remark applies to metric tt*-bundles and symplectic tt^*-bundles. In particular, setting $\theta = \pi$ we find, that $(E, D, -S)$ is a tt*-bundle.

As for tt^*-bundles (E, D, S) over a complex manifold (M, J) we find explicit equations for D and S.

Proposition 1. *Let E be a real vector bundle over an almost complex manifold (M, J) endowed with a connection D and a section $S \in \Gamma(T^*M \otimes \operatorname{End} E)$.*
Then (E, D, S) is a tt-bundle if and only if D and S satisfy the following equations:*

$$R^D + S \wedge S = 0, \tag{2.6}$$

$$S \wedge S \quad \text{is of type } (1, 1), \tag{2.7}$$

$$[D_X, S_Y] - [D_Y, S_X] - S_{[X,Y]} = 0, \quad \forall X, Y \in \Gamma(TM), \tag{2.8}$$

$$[D_X, S_{JY}] - [D_Y, S_{JX}] - S_{J[X,Y]} = 0, \quad \forall X, Y \in \Gamma(TM). \tag{2.9}$$

Fixing a torsion-free connection on (M, J) the last two equations are equivalent to

$$d^D S = 0 \quad \text{and} \quad d^D S_J = 0. \tag{2.10}$$

Proof (*Compare [4] and [17]*). As for tt^*-bundles over complex manifolds (M, J) one calculates using the theorems of addition $2\cos\theta \sin\theta = \sin 2\theta$, $2\cos^2\theta = 1 + \cos 2\theta$ and $2\sin^2\theta = 1 - \cos 2\theta$, the (finite) Fourier decomposition of R^θ in the variable θ. The tt*-equation $R^\theta = 0$ means the vanishing of all Fourier-components. This yields the claimed equations. □

3. Solutions on the tangent bundle of an almost complex manifold

Given an almost complex manifold (M, J) with a flat connection ∇ it is natural to consider the one-parameter family ∇^θ of connections, which is defined by

$$\nabla^\theta_X Y = \exp(\theta J) \nabla_X (\exp(-\theta J) Y) \quad \text{for } X, Y \in \Gamma(TM), \tag{3.1}$$

where $\exp(\theta J) = \cos(\theta) Id + \sin(\theta) J$.
Since ∇ is flat, ∇^θ is flat, too.

We are now going to analyze the form of tt^*-bundles (TM, D, S) for which the connection D^θ defined in Eq. (2.2) is linear equivalent (compare Definition 1) to the connection ∇^θ defined in Eq. (3.1).

Proposition 2. *Given an almost complex manifold (M, J) with a flat connection ∇ and a decomposition of $\nabla = D + S$ into a connection D and a section S in $T^*M \otimes \operatorname{End}(TM)$. Then (TM, D, S) defines a tt^*-bundle, such that D^θ is linear equivalent to ∇^θ with factor $\alpha = \pm 2$ if and only if S and D satisfy*

$$S_{JX} = \pm J S_X Y$$

and

$$-(D_X J) Y = J S_X Y + S_X J Y =: \{S_X, J\} Y$$

for all $X, Y \in \Gamma(TM)$.

Proof. First one has to calculate ∇^θ for $X, Y \in \Gamma(TM)$

$$\begin{aligned} \nabla^\theta_X Y &= \exp(\theta J)(D_X + S_X)(\cos(\theta) Id - \sin(\theta) J) Y \\ &= D_X Y - \exp(\theta J) \sin(\theta) (D_X J) Y + (\cos(\theta) Id + \sin(\theta) J) S_X (\cos(\theta) Id - \sin(\theta) J) Y \\ &= D_X Y - (\cos(\theta)\sin(\theta) + \sin^2(\theta) J)(D_X J) Y + \cos^2(\theta) S_X Y - \sin^2(\theta) J S_X J Y - \cos(\theta)\sin(\theta) [S_X, J] Y, \end{aligned}$$

which yields with the theorems of addition, i.e.

$$2\sin(\theta)\cos(\theta)=\sin(2\theta),\qquad 2\cos^2(\theta)=1+\cos(2\theta)\quad\text{and}\quad 2\sin^2(\theta)=1-\cos(2\theta),$$

the identity

$$\begin{aligned}\nabla^\theta_X Y &= D_X Y-\frac{1}{2}\sin(2\theta)(D_X J)Y-\frac{1}{2}(1-\cos(2\theta))J(D_X J)Y\\ &\quad+\frac{1}{2}(1+\cos(2\theta))S_X Y-\frac{1}{2}(1-\cos(2\theta))JS_X JY-\frac{1}{2}\sin(2\theta)[S_X,J]Y\\ &= D_X Y+\frac{1}{2}[S_X-JS_X J-JD_X J]\,Y\\ &\quad+\frac{1}{2}\sin(2\theta)\,[[J,S_X]-D_X J]\,Y+\frac{1}{2}\cos(2\theta)\,[S_X+JS_X J+JD_X J]\,Y\\ &\overset{!}{=} D_X Y+\cos(\vartheta)T_X Y+\sin(\vartheta)T_{JX}Y\quad\text{with }\vartheta=\pm 2\theta,\end{aligned}$$

where we have to determine $T\in\Gamma(T^*M\otimes\mathrm{End}(TM))$.
Comparing Fourier-coefficients gives

$$J(D_X J)Y=S_X Y-JS_X JY,\quad\text{or equivalently}\tag{3.2}$$

$$-(D_X J)Y=JS_X Y+S_X JY=\{S_X,J\}Y,$$

$$T_X Y=\frac{1}{2}(S_X Y+JS_X JY+J(D_X J)Y)\overset{(3.2)}{=}S_X Y,\tag{3.3}$$

$$\begin{aligned}T_{JX}Y&=\pm\frac{1}{2}([J,S_X]Y-(D_X J)Y)\\ &\overset{(3.2)}{=}\pm\frac{1}{2}(JS_X Y-S_X JY+JS_X Y+S_X JY)=\pm JS_X Y.\end{aligned}\tag{3.4}$$

The last two equations yield the constraint on S

$$S_{JX}=\pm JS_X Y$$

and the first equation the one on D and S. □

We suppose now the connection D to be complex, i.e. $DJ=0$. Such a connection exists on every almost complex manifold (compare [11]).

Corollary 1. *Given an almost complex manifold (M,J) with a flat connection ∇ and a decomposition of $\nabla=D+S$ in a connection D and a section S in $T^*M\otimes\mathrm{End}(TM)$, such that J is D-parallel, i.e. $DJ=0$. Then (TM,D,S) defines a tt^*-bundle, such that D^θ is linear equivalent to ∇^θ with factor $\alpha=\pm 2$ if and only if S satisfies*

$$S_{JX}=\pm JS_X\quad\textit{and}\quad\{S_X,J\}=0.$$

Proof. The second constraint in Proposition 2 is for $DJ=0$ the condition $\{S_X,J\}=0$. The first constraint of Proposition 2 is exactly $S_{JX}=\pm JS_X\overset{\{S_X,J\}=0}{=}\mp S_X J$. □

We are now going to show some uniqueness results. Therefore we prove the

Lemma 1. *Let (M,J) be an almost complex manifold. Given a connection ∇ on M which decomposes as $\nabla=D+S$, where D is a connection on M and S is a section in $T^*M\otimes\mathrm{End}(TM)$, such that J is D-parallel, i.e. $DJ=0$ and S anticommutes with J, i.e. $\{S_X,J\}=0$ for all $X\in\Gamma(TM)$. Then S and D are uniquely given by*

$$S_X Y=\frac{1}{2}J(\nabla_X J)Y\quad\textit{and}\quad D_X Y=\nabla_X Y-S_X Y\quad\textit{for }X,Y\in\Gamma(TM).\tag{3.5}$$

Otherwise, given a connection ∇ and define D and S by Eq. (3.5), then D and S satisfy $DJ=0$ and $\{S_X,J\}=0$.

Proof. First we observe: $\nabla = D + S$ and $S_X JY = \frac{1}{2}J(\nabla_X J)JY = -\frac{1}{2}J^2(\nabla_X J)Y = -JS_X Y$, where the second equality follows from deriving $J^2 = -Id$. Further it is

$$(D_X J)Y = (\nabla_X J)Y - [S_X, J] \overset{\{S_X, J\}=0}{=} (\nabla_X J)Y + 2JS_X = 0.$$

Now we prove the uniqueness: Suppose there exist D' and S' with the same properties. Then we get

$$0 = (D'_X J)Y = (\nabla_X J)Y - [S'_X, J]Y = (\nabla_X J)Y + 2JS'_X Y$$

and consequently

$$S'_X Y = \frac{1}{2}J(\nabla_X J)Y = S_X Y \quad \text{and} \quad D'_X Y = \nabla_X Y - S'_X Y = \nabla_X Y - S_X Y = D_X Y. \quad \square$$

Summarizing Corollary 1 and Lemma 1 we find the following uniqueness result:

Theorem 1. *Given an almost complex manifold (M, J) with a flat connection ∇ and a decomposition of $\nabla = D + S$ in a connection D and a section S in $T^*M \otimes \mathrm{End}(TM)$, such that J is D-parallel, i.e. $DJ = 0$. If (TM, D, S) defines a tt^*-bundle, such that D^θ is linear equivalent to ∇^θ with factor $\alpha = \pm 2$, then D and S are uniquely determined by $S = \frac{1}{2}J(\nabla J)$ and $D = \nabla - S$.*

Moreover, (TM, D, S) as above defines a tt^*-bundle, such that D^θ is linear equivalent to ∇^θ with factor $\alpha = \pm 2$, if and only if J satisfies $(\nabla_{JX} J) = \pm J(\nabla_X J)$ and D and S are given by $S = \frac{1}{2}J(\nabla J)$ and $D = \nabla - S$.

Now we are going to give some classes of examples which satisfy the condition $S_{JX} = \pm JS_X$.

Proposition 3. *Given an almost complex manifold (M, J) with a connection ∇ and let S be the section in $T^*M \otimes \mathrm{End}(TM)$ defined by*

$$S := \frac{1}{2}J(\nabla J). \tag{3.6}$$

If the pair (∇, J) satisfies one of the following conditions

(i) *(∇, J) is* special, *i.e. $(\nabla_X J)Y = (\nabla_Y J)X$ for all $X, Y \in \Gamma(TM)$,*
(ii) *(∇, J) satisfies the* nearly Kähler condition, *i.e. $(\nabla_X J)Y = -(\nabla_Y J)X$ for all $X, Y \in \Gamma(TM)$,*

then it holds that $S_{JX}Y = -JS_X Y$.

Proof. If the condition (i) or (ii) holds, we obtain the identity

$$\begin{aligned}(\nabla_{JX} J)Y &= \pm(\nabla_Y J)JX = \pm[-\nabla_Y X - J\nabla_Y(JX)] \\ &= \mp J[\nabla_Y(JX) - J\nabla_Y X] = \mp J(\nabla_Y J)X = -J(\nabla_X J)Y.\end{aligned}$$

The following calculation finishes the proof

$$S_{JX}Y = \frac{1}{2}J(\nabla_{JX} J)Y = -\frac{1}{2}J^2(\nabla_X J)Y = -JS_X Y. \quad \square$$

Proposition 4. *Given a complex manifold (M, J) with a connection ∇ and let S be the section in $T^*M \otimes \mathrm{End}(TM)$ defined by*

$$S := \frac{1}{2}J(\nabla J). \tag{3.7}$$

If ∇ is (anti-)adapted, *i.e. $\nabla_{JX}Y = \pm J\nabla_X Y$ for all holomorphic vector-fields X, Y, then it holds that $S_{JX}Y = \pm JS_X Y$.*

Proof. From ∇ (anti-)adapted we obtain for all holomorphic vector-fields X, Y:

$$(\nabla_{JX} J)Y = \pm J(\nabla_X J)Y.$$

The following computation gives the proof

$$S_{JX}Y = \frac{1}{2}J(\nabla_{JX}J)Y = \pm\frac{1}{2}J^2(\nabla_X J)Y = \pm J S_X Y. \quad \square$$

Remark 2. One sees easily that condition (i) in Proposition 3 is the symmetry of $S_X Y$ and condition (ii) is its anti-symmetry. We recall that if the connection ∇ is torsion free, flat and special then (M, J, ∇) is a special complex manifold, see [1,9]. tt^*-bundles coming from special complex manifolds and special Kähler manifolds were studied in [4].
Further we want to remark that the second condition in Proposition 3 arises in nearly Kählerian geometry and therefore is quite natural. These two geometries as solutions of tt^*-geometry are discussed later in this work.
Finally, the notion of adapted connections appeared in the study of decompositions on (para-holomorphic) vector bundles, compare for example [2] for the complex and [13] for the para-complex case.

4. Solutions on almost hermitian manifolds

In this section we consider almost complex manifolds (M, J) endowed with a flat connection ∇ such that (∇, J) is special or satisfies the nearly Kähler condition and analyze under which additional assumptions these define symplectic or metric tt^*-bundles.

Definition 3. An almost complex manifold (M, J) is called almost hermitian if there exists a pseudo-Riemannian metric g which is hermitian, i.e. it satifies $J^*g(\cdot,\cdot) = g(J\cdot, J\cdot) = g(\cdot,\cdot)$.

First, we recall a lemma from tensor-algebra:

Lemma 2. *Let V be a vector-space, $\alpha \in T^3(V^*)$ an element in the third tensorial power of V^*, the dual space of V. Suppose that $\alpha(X, Y, Z)$ is symmetric (resp. anti-symmetric) in X, Y and Y, Z and $\alpha(X, Y, Z)$ is anti-symmetric (resp. symmetric) in X, Z then $\alpha = 0$.*

Proof. It is $\alpha(X, Y, Z) = \epsilon\alpha(Y, X, Z) = \epsilon\alpha(X, Z, Y)$ with $\epsilon \in \{\pm 1\}$ which implies $\alpha(X, Y, Z) = \epsilon\alpha(Y, X, Z) = \epsilon^2\alpha(Y, Z, X) = \epsilon^3\alpha(Z, Y, X)$. But further it holds $\alpha(X, Y, Z) = -\epsilon\alpha(Z, Y, X)$ and consequently $-\alpha(Z, Y, X) = \epsilon^2\alpha(Z, Y, X) = \alpha(Z, Y, X)$. This shows $\alpha = 0$. $\square$

The subsequent proposition shows that the condition to be special is not compatible with symplectic tt^*-bundles:

Proposition 5. *Given an almost hermitian manifold (M, J, g) with a flat connection ∇, such that (∇, J) is special. Define S, a section in $T^*M \otimes \mathrm{End}(TM)$ by*

$$S := \frac{1}{2}J(\nabla J), \tag{4.1}$$

then $(TM, D = \nabla - S, S)$ defines a tt^*-bundle. Suppose, that $(TM, D, S, \omega = g(J\cdot,\cdot))$ is a symplectic tt^*-bundle, then it is trivial, i.e. $S = 0$.

Proof. In fact we know from Theorem 1 and Proposition 3, that (TM, D, S) is a tt^*-bundle. Suppose, that $(TM, D, S, \omega = g(J\cdot,\cdot))$ is a symplectic tt^*-bundle. To finish the proof, we define the tensor

$$\alpha(X, Y, Z) := \omega(S_X Y, Z) = g(J S_X Y, Z).$$

$\alpha(X, Y, Z)$ is symmetric in X, Y, since (∇, J) is special, i.e. ∇J is symmetric in X, Y. Further it holds that

$$\begin{aligned} \alpha(X, Y, Z) &= \omega(S_X Y, Z) = -\omega(Z, S_X Y) \\ &= -\omega(Z, S_Y X) = -\omega(S_Y Z, X) = -\omega(S_Z Y, X) = -\alpha(Z, Y, X) \end{aligned}$$

which is the anti-symmetry of $\alpha(X, Y, Z)$ in X, Z. Finally

$$\begin{aligned} \alpha(X, Y, Z) &= \omega(S_X Y, Z) = \omega(Y, S_X Z) \\ &= \omega(Y, S_Z X) = -\omega(S_Z X, Y) = -\alpha(Z, X, Y) = -\alpha(X, Z, Y), \end{aligned}$$

i.e. the anti-symmetry of $\alpha(X, Y, Z)$ in Y, Z.
Hence α vanishes and consequently S. $\square$

Otherwise, the nearly Kähler condition is not compatible with metric tt^*-bundles:

Proposition 6. *Given an almost hermitian manifold (M, J, g) with a flat connection ∇, such that (∇, J) satisfies the nearly Kähler condition. Define S, a section in $T^*M \otimes \mathrm{End}(TM)$ by*

$$S := \frac{1}{2} J(\nabla J), \tag{4.2}$$

then $(TM, D = \nabla - S, S)$ defines a tt^*-bundle. Suppose, that (TM, D, S, g) is a metric tt^*-bundle, then it is trivial, i.e. $S = 0$.

Proof. In fact we know from Theorem 1 and Proposition 3, that (TM, D, S) is a tt^*-bundle. Suppose, that it is a metric tt^*-bundle. To finish the proof, we define the tensor

$$\alpha(X, Y, Z) := g(S_X Y, Z).$$

$\alpha(X, Y, Z)$ is anti-symmetric in X, Y, since, by the nearly Kähler condition, ∇J is anti-symmetric in X, Y. Further it holds that

$$\begin{aligned}\alpha(X, Y, Z) &= g(S_X Y, Z) = g(Z, S_X Y)\\ &= -g(Z, S_Y X) = -g(S_Y Z, X) = g(S_Z Y, X) = \alpha(Z, Y, X)\end{aligned}$$

which is the symmetry of $\alpha(X, Y, Z)$ in X, Z. Finally

$$\begin{aligned}\alpha(X, Y, Z) &= g(S_X Y, Z) = g(Y, S_X Z)\\ &= -g(Y, S_Z X) = -g(S_Z X, Y) = -\alpha(Z, X, Y) = \alpha(X, Z, Y),\end{aligned}$$

i.e. the symmetry of $\alpha(X, Y, Z)$ in Y, Z.
Hence α vanishes by the above lemma and so does S. □

This theorem gives solutions of symplectic tt^*-bundles on the tangent bundle, which are more general then the later discussed nearly Kähler manifolds in the sense, that we admit connections ∇ having torsion, but more special in the sense, that our connection ∇ has to be flat:

Theorem 2. *Given an almost hermitian manifold (M, J, g) with a flat metric connection ∇, such that (∇, J) satisfies the nearly Kähler condition. Define S, a section in $T^*M \otimes \mathrm{End}(TM)$ by*

$$S := \frac{1}{2} J(\nabla J), \tag{4.3}$$

then $(TM, D = \nabla - S, S, \omega = g(J\cdot, \cdot))$ defines a symplectic tt^*-bundle. Moreover, it holds $DJ = 0$ and $T^D = T^\nabla - 2S$.

Proof. In fact we know from Theorem 1 and Proposition 3, that (TM, D, S) is a tt^*-bundle. It remains to check that $D\omega = 0$ and that S is ω-symmetric.
First we remark, that, since g is hermitian and $\nabla g = 0$, $\nabla_X J$ is skew-symmetric with respect to g. Using this we show by the following calculation, that S is skew-symmetric with respect to g:

$$\begin{aligned}2g(S_X Y, Z) &= g(J(\nabla_X J)Y, Z) = -g((\nabla_X J)Y, JZ)\\ &= g(Y, (\nabla_X J)JZ) = -g(Z, J(\nabla_X J)Y) = -2g(Y, S_X Z).\end{aligned}$$

The definition of $\omega = g(J\cdot, \cdot)$ and $\{S_X, J\} = 0$ yield the ω-symmetry of S_X. Further it holds that $D = \nabla - \frac{1}{2} J\nabla J$, which implies

$$DJ = \nabla J - \frac{1}{2}[J\nabla J, J] = 0.$$

This proves $D\omega = 0$ if and only if $Dg = 0$. But $\nabla g = 0$ and S is skew-symmetric with respect to g, so g is parallel for $D = \nabla - S$. This shows that $(TM, D = \nabla - S, S, \omega)$ is a symplectic tt^*-bundle.
Calculating the torsion we find $T^D(X, Y) = T^\nabla(X, Y) - S_X Y + S_Y X = T^\nabla(X, Y) - 2S_X Y$. □

We recall the definition of special complex and special Kähler manifolds of [1,9]:

Definition 4. A special Kähler manifold consists of the data (M, J, g, ∇) where (M, J, g) is a Kähler manifold with Kähler-form ω satisfying $\nabla\omega = 0$ and (M, J, ∇) is a special complex manifold, i.e. (M, J) is a complex manifold endowed with a flat and torsion-free connection ∇ such that (∇, J) is special.

The following theorem gives solutions of metric tt^*-bundles on the tangent bundle, which are more general than special Kähler manifolds in the sense, that we admit connections ∇ with torsion.

Theorem 3. *Given an almost hermitian manifold (M, J, g) with a flat connection ∇, such that (∇, J) is special and the two-form $\omega = g(J\cdot, \cdot)$ is ∇-parallel. Define S, a section in $T^*M \otimes \mathrm{End}(TM)$, by*

$$S := \frac{1}{2} J(\nabla J), \tag{4.4}$$

then $(TM, D = \nabla - S, S, g)$ defines a metric tt^*-bundle. Moreover, it holds that $DJ = 0$ and $T^D = T^\nabla$.
Suppose, that ∇ is torsion free, then D is the Levi-Civita connection of g, (M, J, g) is a Kähler manifold and (M, J, g, ∇) is a special Kähler manifold.

Proof. In fact we know from Theorem 1 and Proposition 3, that (TM, D, S) is a tt^*-bundle.
It remains to check $Dg = 0$ and that S is g-symmetric.

First we remark that $\omega(JX, Y) = -\omega(X, JY)$ as g is hermitian. This yields using $\nabla\omega = 0$ the ω-skew-symmetry of $\nabla_X J$, which implies that $S_X = \frac{1}{2} J(\nabla J)$ is ω-skew-symmetric, since $J(\nabla_X J) = -(\nabla_X J)J$. Finally $\{S_X, J\} = 0$ shows the g-symmetry of S_X.
Further it is

$$DJ = \nabla J - \frac{1}{2}[J\nabla J, J] = 0$$

and consequently $Dg = 0$ is equivalent to $D\omega = 0$.
From $\nabla\omega = 0$ and the ω-skew-symmetry of S it follows, that $D\omega = (\nabla - S)\omega = 0$.
The symmetry of ∇J, i.e. $(\nabla_X J)Y = (\nabla_Y J)X$ for all $X, Y \in TM$ implies $S_X Y = S_Y X$ and consequently $T^D = T^\nabla$.
Suppose now that ∇ is torsion free. This shows, that $D = \nabla - S$ is torsion free and consequently the Levi-Civita connection of g. Further the equation $\nabla\omega = 0$ implies $d\omega = 0$ since ∇ is torsion free. Hence (M, J, g) is Kähler. In addition (M, J, ∇) is special complex by the conditions on ∇ and J. Therefore (M, J, g, ∇) is a special Kähler manifold, as it holds $\nabla\omega = 0$. □

In [4] we studied special Kähler solutions of tt^*-geometry in more detail.
Now we want to apply the above results to nearly Kähler manifolds. In order to do this we recall some notions and results of nearly Kähler geometry (compare for example Friedrich [10] and Nagy [15,16]):

Definition 5. An almost hermitian manifold (M, J, g) is called a nearly Kähler manifold, if its Levi-Civita connection $\nabla = \nabla^g$ satisfies the equation

$$(\nabla_X J)Y = -(\nabla_Y J)X, \quad \forall\, X, Y \in \Gamma(TM). \tag{4.5}$$

A nearly Kähler manifold is called strict, if $\nabla J \neq 0$.

We recall that the tensor ∇J defines two three-forms A, B

$$A(X, Y, Z) := g((\nabla_X J)Y, Z) \quad \text{and} \quad B(X, Y, Z) := g((\nabla_X J)Y, JZ) \quad \text{with } X, Y, Z \in TM,$$

which are both real three-forms of type $(3, 0) + (0, 3)$.
A connection of particular importance in nearly Kähler geometry is the connection $\overline{\nabla}$ defined by

$$\overline{\nabla}_X Y := \nabla_X Y + \frac{1}{2}(\nabla_X J)JY, \quad \text{for all } X, Y \in \Gamma(TM). \tag{4.6}$$

We may remark, that $\overline{\nabla}$ is the unique connection with totally skew-symmetric torsion (compare [10]).

The torsion of the connection $\overline{\nabla}$ is given by

$$T^{\overline{\nabla}}(X, Y) = (\nabla_X J)JY, \quad \text{for all } X, Y \in \Gamma(TM) \tag{4.7}$$

and it vanishes if and only if (M, J, g) is a Kähler manifold.

Corollary 2. *Given a nearly Kähler manifold (M, J, g) such that its Levi-Civita connection ∇ is flat and let S be the section in $T^*M \otimes \mathrm{End}(TM)$ defined by*

$$S := \frac{1}{2}J(\nabla J) \tag{4.8}$$

then $(TM, \overline{\nabla}, S)$ defines a tt-bundle. Suppose, that $(TM, \overline{\nabla}, S, g)$ is a metric tt*-bundle, then it is trivial, i.e. $S = 0$ and consequently (M, J, g) is Kähler.*

Proof. By setting $D = \overline{\nabla}$ we are in the situation of Proposition 6. □

Theorem 4. *Given a nearly Kähler manifold (M, J, g) such that its Levi-Civita connection ∇ is flat. Let S be the section in $T^*M \otimes \mathrm{End}(TM)$ defined by*

$$S := \frac{1}{2}J(\nabla J), \tag{4.9}$$

then $(TM, \overline{\nabla}, S, \omega := g(J\cdot, \cdot))$ is a symplectic tt-bundle. Further it holds*

$$B(X, Y, Z) = -2g(S_X Y, Z) \quad \text{and} \quad \overline{\nabla} J = 0. \tag{4.10}$$

Proof. By setting $D = \overline{\nabla}$ we are in the situation of Theorem 2. In addition it holds that

$$2g(S_X Y, Z) = g(J(\nabla_X J)Y, Z) = -B(X, Y, Z). \quad \square$$

A constructive classification of nearly Kähler manifolds with flat Levi-Civita connection was given in a common paper [5] with V. Cortés.

5. Pluriharmonic maps from almost complex manifolds into pseudo-Riemannian manifolds

In this section we generalize the notion of a pluriharmonic map to maps from almost complex manifolds to pseudo-Riemannian manifolds. Afterwards we show that maps admitting a generalisation of an associated family (compare [8]) give rise to a pluriharmonic map and we give conditions under which a pluriharmonic map is harmonic. Let (M, J) be an almost complex manifold of real dimension $2n$. It is well-known (compare [11]) that on every almost complex manifold there exists a complex connection with torsion $T = \frac{1}{4}N_J$ where

$$N_J(X, Y) = [JX, JY] - [X, Y] - J[X, JY] - J[JX, Y]$$

is the Nijenhuis[2] tensor of J.

Definition 6. Let (M, J) be an almost complex manifold. A connection D on the tangent bundle of M is called nice if it is complex and its torsion satisfies $4T = N_J$.

As the reader may check all statements of this section rest true by replacing the condition $4T = N_J$ by $-4T = N_J$. Next, we introduce the notion of a pluriharmonic map from an almost complex manifold:

Definition 7. Let (M, J, D) be an almost complex manifold endowed with a nice connection D on TM and N a smooth manifold endowed with a connection ∇^N. Denote by ∇ the connection on $T^*M \otimes f^*TN$ which is induced by D and ∇^N.

[2] In [11] the Nijenhuis tensor is defined with a factor 2.

A smooth map $f : M \to N$ is pluriharmonic if and only if it satisfies the equation

$$(\nabla \mathrm{d} f)^{1,1} = 0. \tag{5.1}$$

Remark 3. We may remark, that for a complex manifold (M, J) and a pseudo-Riemannian target manifold (N, h) with its Levi-Civita connection ∇^h the pluriharmonic equation (5.1) does not depend on the connection D if D is chosen in an appropriate class (compare [4]). In fact nice connections on complex manifolds belong to this class. A very often considered case is Kähler manifolds (M, J, g), where D is taken to be the Levi-Civita connection.

As preparation for associated families we recall an integrability condition satisfied by the differential of a smooth map. Let N be a smooth manifold with a connection ∇^N on its tangent bundle having torsion tensor T^N. Given a second smooth manifold M and a smooth map $f : M \to N$, the differential $F := \mathrm{d} f : TM \to f^*TN = E$ induces a vector bundle homomorphism between the tangent bundle of M and the pull-back of TN via f. The torsion T^N of N induces a bundle homomorphism $T^E : \Lambda^2 E \to E$ satisfying the identity

$$\nabla^E_V F(W) - \nabla^E_W F(V) - F([V, W]) = T^E(F(V), F(W)), \tag{5.2}$$

where $\nabla^E = f^*\nabla^N$ denotes the pull-back connection, i.e. the connection which is induced on E by ∇^N and where $V, W \in \Gamma(TM)$.

In the rest of the section we denote by D a nice connection on the almost complex manifold (M, J). Under this assumption we restate the condition (5.2)

$$\begin{aligned} T^E(F(V), F(W)) &= \nabla^E_V F(W) - \nabla^E_W F(V) - F([V, W]) \\ &= \nabla^E_V F(W) - \nabla^E_W F(V) - F(D_V W) + F(D_W V) + F(T(V, W)) \\ &= \nabla^E_V F(W) - \nabla^E_W F(V) - F(D_V W) + F(D_W V) + \frac{1}{4} F(N_J(V, W)) \\ &= (\nabla_V F)W - (\nabla_W F)V + \frac{1}{4} F(N_J(V, W)), \end{aligned} \tag{5.3}$$

where ∇ is the connection induced on $T^*M \otimes E$ by D and ∇^E.

Later in this work we consider the case where N is a pseudo-Riemannian symmetric space with its Levi-Civita connection ∇^N.

Given an angle $\alpha \in [0, 2\pi]$ we define $\mathcal{R}_\alpha : TM \to TM$ as

$$\mathcal{R}_\alpha(X) = \cos(\alpha)X + \sin(\alpha)JX.$$

This defines a parallel endomorphism field on the tangent bundle TM of M. The eigenvalues of which are $e^{\sqrt{-1}\alpha}$ on $T^{1,0}M$ and $e^{-\sqrt{-1}\alpha}$ on $T^{0,1}M$, as one sees easily.

An associated family for f is a family of maps $f_\alpha : M \to N, \alpha \in [0, 2\pi]$, such that

$$\Phi_\alpha \circ \mathrm{d} f_\alpha = \mathrm{d} f \circ \mathcal{R}_\alpha, \quad \forall \alpha \in \mathbb{R}, \tag{5.4}$$

for some bundle isomorphism $\Phi_\alpha : f_\alpha^* TN \to f^*TN, \alpha \in \mathbb{R}$, which is parallel with respect to ∇^N in the sense that

$$\Phi_\alpha \circ (f_\alpha^* \nabla^N) = (f^* \nabla^N) \circ \Phi_\alpha.$$

One observes, that each map f_α of an associated family itself admits an associated family.

Theorem 5. *Let (M, J) be an almost complex manifold endowed with a nice connection D, N a smooth manifold with a torsion-free connection ∇^N and $f : (M, D, J) \to (N, \nabla^N)$ a smooth map admitting an associated family f_α, then f is pluriharmonic. More precisely, each map of the associated family f_α is pluriharmonic.*

Proof. As Φ_α is parallel with respect to ∇^N, ∇^N is torsion-free and D is nice, we can apply Eq. (5.3) to the family $\mathrm{d} f_\alpha = F_\alpha = \Phi_\alpha^{-1} \circ \mathrm{d} f \circ \mathcal{R}_\alpha$ to obtain

$$(\nabla_V F_\alpha)W - (\nabla_W F_\alpha)V + \frac{1}{4} F_\alpha(N_J(V, W)) = 0.$$

Since $\mathcal{R}_\alpha$ is D-parallel we obtain

$$(\nabla_X F_\alpha) = \Phi_\alpha^{-1} \circ (\nabla_X F) \circ \mathcal{R}_\alpha.$$

If $Z = X + \mathrm{i}JX$ and $W = Y - \mathrm{i}JY$ have different type it holds $N_J(Z, W) = 0$, where we extended the Nijenhuis tensor complex linearly. This implies

$$(\nabla_V F_\alpha)W = (\nabla_W F_\alpha)V, \quad \forall \alpha \in [0, 2\pi]$$

and using this we obtain

$$\begin{aligned}(\nabla_Z F_\alpha)W &= e^{\sqrt{-1}\alpha}\, \Phi_\alpha^{-1}(\nabla_Z F)W\\ (\nabla_W F_\alpha)Z &= e^{-\sqrt{-1}\alpha}\, \Phi_\alpha^{-1}(\nabla_W F)Z = e^{-\sqrt{-1}\alpha}\, \Phi_\alpha^{-1}(\nabla_Z F)W\end{aligned}$$

for all $\alpha \in [0, 2\pi]$. Since this should coincide, it follows that $(\nabla \mathrm{d} f)^{(1,1)} = 0$, i.e. $f : (M, D, J) \to (N, \nabla^N)$ is pluriharmonic. The rest follows, since each map of the associated family f_α admits an associated family $g_\beta = f_{(\alpha+\beta) \bmod 2\pi}$. □

This motivates the definition

Definition 8. Let (M, J) be an almost complex manifold endowed with a nice connection D, N a smooth manifold endowed with a torsion-free connection ∇^N. A smooth map $f : (M, D, J) \to (N, \nabla^N)$ is said to be $\mathbb{S}^1$-pluriharmonic if and only if it admits an associated family.

Given a hermitian metric g on M then in general a nice connection D is not the Levi-Civita connection ∇^g of g. Therefore the pluriharmonic Eq. (5.1) does not imply the harmonicity of f. But if the tensor $D - \nabla^g$ is trace-free the pluriharmonic equation implies the harmonic equation. This is true in the case of a special Kähler manifold (M, J, g, ∇) and for a nearly Kähler manifold, where $D = \overline{\nabla}$ and $\overline{\nabla} - \nabla^g$ is skew-symmetric.

Proposition 7. *Let (M, J, g) be an almost hermitian manifold endowed with a nice connection D, N a pseudo-Riemannian manifold with its Levi-Civita connection ∇^N. Suppose that the tensor $S = \nabla^g - D$ is trace free. Then a pluriharmonic map $f : M \to N$ is harmonic.*

Proof. We consider

$$\begin{aligned}\mathrm{tr}_g(\nabla \mathrm{d} f) &= \sum_i g(e_i, e_i)[\nabla^E_{e_i} \mathrm{d} f(e_i) - \mathrm{d} f(D_{e_i} e_i)]\\ &= \sum_i g(e_i, e_i)[\nabla^E_{e_i} \mathrm{d} f(e_i) - \mathrm{d} f((\nabla^g - S)_{e_i} e_i)]\\ &= \sum_i g(e_i, e_i)[\nabla^E_{e_i} \mathrm{d} f(e_i) - \mathrm{d} f(\nabla^g_{e_i} e_i)]\\ &= \mathrm{tr}_g(\tilde{\nabla}^g \mathrm{d} f)\end{aligned}$$

where $\tilde{\nabla}^g$ is the connection induced on $T^*M \otimes E$ by ∇^g and ∇^E and e_i is an orthogonal basis for g on TM. But from the pluriharmonic equation and since g is hermitian we obtain

$$\mathrm{tr}_g(\nabla \mathrm{d} f) = \mathrm{tr}_g(\nabla \mathrm{d} f^{(1,1)}) = 0. \quad \square$$

6. Related pluriharmonic and harmonic maps

6.1. The classifying map of a flat nearly Kähler manifold

In this section we consider simply connected almost hermitian manifolds (M, J, g) endowed with a flat metric connection ∇ such that (∇, J) satisfies the nearly Kähler condition.
In particular, simply connected flat nearly Kähler manifolds (M^{2n}, J, g), i.e. nearly Kähler manifolds (M, J, g) with flat Levi-Civita connection ∇^g are of this type. Since (M, g, ∇) is simply connected and flat, we may identify by

fixing a ∇-parallel frame s_0 its tangent bundle TM with $(M \times V, \langle \cdot, \cdot \rangle)$, where $V = \mathbb{C}^n = (\mathbb{R}^{2n}, j_0)$ is endowed with the standard scalar product $\langle \cdot, \cdot \rangle$ of the same hermitian signature (p, q) as the hermitian metric g.
The compatible complex structure J defines via this identification a map

$$J : M \to \mathfrak{J}(V, \langle \cdot, \cdot \rangle),$$

where $\mathfrak{J}(V, \langle \cdot, \cdot \rangle)$ is the set of complex structures on V which are compatible with $\langle \cdot, \cdot \rangle$ and the orientation of $V = \mathbb{R}^{2n}$.
We shortly explain the differential geometry of this set:
One can consider $\mathfrak{J}(V, \langle \cdot, \cdot \rangle)$ as a subset in the vector space $\mathfrak{so}(2p, 2q) = \mathfrak{so}(V) \subset \mathrm{Mat}(\mathbb{R}^{2n})$ characterized by the set of $n(2n+1)$ equations

$$f(j) = -\mathbb{1}_{2n}, \tag{6.1}$$

where $f : \mathrm{Mat}(\mathbb{R}^{2n}) \to \mathrm{Mat}(\mathbb{R}^{2n})$ is given by $f : A \mapsto A^2$. The differential of this map is $\mathrm{d}f_A(H) = \{A, H\}$ for $A, H \in \mathrm{Mat}(\mathbb{R}^{2n})$. In addition, $\mathrm{d}f$ has constant rank in points j satisfying Eq. (6.1), since one sees

$$\ker \mathrm{d}f_j = \{A \in \mathfrak{so}(V) \mid \{j, A\} = 0\},$$
$$\mathrm{im}\, \mathrm{d}f_j \cong \{A \in \mathfrak{so}(V) \mid [j, A] = 0\} \cong \mathfrak{u}(p, q).$$

Applying the regular value theorem $\mathfrak{J}(V, \langle \cdot, \cdot \rangle)$ is shown to be a submanifold of $\mathfrak{so}(V)$. Its tangent space at $j \in \mathfrak{J}(V, \langle \cdot, \cdot \rangle)$ is

$$T_j \mathfrak{J}(V, \langle \cdot, \cdot \rangle) = \ker \mathrm{d}f_j = \{A \in \mathfrak{so}(V) \mid \{j, A\} = 0\}. \tag{6.2}$$

Moreover, $\mathfrak{J}(V, \langle \cdot, \cdot \rangle)$ can be identified with the pseudo-Riemannian symmetric space $SO_0(2p, 2q)/U(p, q)$, where $SO_0(2p, 2q)$ is the identity component of the special pseudo-orthogonal group $SO(2p, 2q)$ and $U(p, q)$ is the unitary group of signature (p, q), by the map

$$\Phi : SO_0(2p, 2q)/U(p, q) \to \mathfrak{J}(V, \langle \cdot, \cdot \rangle),$$
$$gK \mapsto g\, j_0\, g^{-1},$$

which maps the canonical base point $o = eK$ to j_0.
Any $j \in \mathfrak{J}(V, \langle \cdot, \cdot \rangle)$ defines a symmetric decomposition of $\mathfrak{so}(V)$ by

$$\mathfrak{p}(j) = \{A \in \mathfrak{so}(V) \mid \{j, A\} = 0\},$$
$$\mathfrak{k}(j) = \{A \in \mathfrak{so}(V) \mid [j, A] = 0\} \cong \mathfrak{u}(p, q).$$

In particular $\mathfrak{k}(j_0) = \mathfrak{u}(p, q)$. Moreover, one observes $T_j \mathfrak{J}(V, \langle \cdot, \cdot \rangle) = \mathfrak{p}(j)$.
Let $\tilde{j} \in SO_0(2p, 2q)/U(p, q)$ and $j = \Phi(\tilde{j})$, then $T_{\tilde{j}} SO_0(2p, 2q)/U(p, q)$ is canonically identified with $\mathfrak{p}(j)$. We determine now the differential of the above identification.

Proposition 8. *Let $\Psi = \Phi^{-1} : \mathfrak{J}(V, \langle \cdot, \cdot \rangle) \to SO_0(2p, 2q)/U(p, q)$. Then it holds at $j \in \mathfrak{J}(V, \langle \cdot, \cdot \rangle)$*

$$\mathrm{d}\Psi : T_j \mathfrak{J}(V, \langle \cdot, \cdot \rangle) \ni X \mapsto -\frac{1}{2} j^{-1} X \in \mathfrak{p}(j). \tag{6.3}$$

This can be used to relate the differential of a map

$$J : M \to \mathfrak{J}(V, \langle \cdot, \cdot \rangle)$$

and a map

$$\tilde{J} = \Psi \circ J : M \to SO_0(2p, 2q)/U(p, q)$$

by

$$\mathrm{d}\tilde{J} = -\frac{1}{2} J^{-1} \mathrm{d}J.$$

We remember that under the above assumptions $(TM, D = \nabla - S, S = \frac{1}{2} J(\nabla J), \omega = g(J\cdot, \cdot))$ defines a symplectic tt^*-bundle.

Theorem 6. *Let (M, J, g) be a simply connected almost hermitian manifold endowed with a flat metric connection ∇ such that (∇, J) satisfies the nearly Kähler condition, then $(TM, D = \nabla - S, S = \frac{1}{2}J(\nabla J), \omega = g(J\cdot, \cdot))$ defines a symplectic tt^*-bundle and the matrix of J in a D^θ-flat frame $s^\theta = (s_i^\theta)$ defines an $\mathbb{S}^1$-pluriharmonic map $\tilde{J}^\theta : M \to \mathcal{J}(V, \langle\cdot,\cdot\rangle) \to SO_0(2p, 2q)/U(p, q)$.*
In particular, given a nice connection D on M the map $\tilde{J}^\theta : (M, J, D) \to SO_0(2p, 2q)/U(p, q)$ is pluriharmonic.

Proof. We observe $D^\theta g = 0$ since $\nabla g = 0$ and $S_X^\theta := \cos(\theta)S_X + \sin(\theta)S_{JX}$ takes values in $\mathfrak{so}(V)$. Therefore we can choose for each θ the D^θ-flat frame s^θ orthonormal, such that $s^{\theta=0} = s_0$. This yields using $DJ = 0$ (compare Theorem 2)

$$X.g(Js_i^\theta, s_j^\theta) = g(D_X^\theta(Js_i^\theta), s_j^\theta) = g((D_X^\theta J)s_i^\theta, s_j^\theta) = g([S_X^\theta, J]s_i^\theta, s_j^\theta) = -2g(JS_X^\theta s_i^\theta, s_j^\theta).$$

Let S^{s^θ}, J^{s^θ} be the representations of S and J in the frame s^θ, then

$$(J^{s^\theta})^{-1}X(J^{s^\theta}) = -2S^{s^\theta}$$

or

$$\mathrm{d}\tilde{J}^\theta = (s^\theta)^{-1} \circ S^\theta \circ s^\theta,$$

where the frame s^θ is seen as a map $s^\theta : M \times V \to TM$. This shows for $X \in \Gamma(TM)$

$$\begin{aligned}\mathrm{d}\tilde{J}^\theta(X) &= (s^\theta)^{-1} \circ S_X^\theta \circ (s^\theta) = (s^\theta)^{-1} \circ S_{\mathcal{R}_\theta X} \circ (s^\theta)\\ &= ((s^\theta)^{-1}s^0) \circ \mathrm{d}\tilde{J}(\mathcal{R}_\theta X) \circ ((s^0)^{-1}s^\theta)\\ &= Ad_{\alpha_\theta}^{-1} \circ \mathrm{d}\tilde{J}(\mathcal{R}_\theta X) = \Phi_\theta^{-1} \circ \mathrm{d}\tilde{J}(\mathcal{R}_\theta X),\end{aligned}$$

where $\alpha_\theta = (s^\theta)^{-1}s^0$ is the frame change from s_0 to s_θ and $\Phi_\theta = Ad_{\alpha_\theta}$ which is parallel with respect to the Levi-Civita connection on $SO_0(2p, 2q)/U(p, q)$. This shows, that $\tilde{J}^\theta$ is $\mathbb{S}^1$-pluriharmonic. Given a nice connection D on M Theorem 5 shows that $\tilde{J}^\theta$ is pluriharmonic. □

We emphasize the nearly Kähler setting:

Corollary 3. *Let (M, J, g) be a flat nearly Kähler manifold and $(TM, \overline{\nabla} = \nabla^g - S, S = \frac{1}{2}J(\nabla J), \omega(\cdot,\cdot) = g(J\cdot, \cdot))$ the associated symplectic tt^*-bundle, then the matrix of J in a D^θ-flat frame $s^\theta = (s_i^\theta)$ defines an $\mathbb{S}^1$-pluriharmonic map $\tilde{J}^\theta : M \to \mathcal{J}(V, \langle\cdot,\cdot\rangle) \to SO_0(2p, 2q)/U(p, q)$.*

For nearly Kähler manifolds we have more precise information about the map $\tilde{J}^\theta$:

Theorem 7. *Let (M, J, g) be a flat nearly Kähler manifold and $(TM, \overline{\nabla} = \nabla^g - S, S = \frac{1}{2}J(\nabla J), \omega(\cdot,\cdot) = g(J\cdot, \cdot))$ the associated symplectic tt^*-bundle. Then the connection $\overline{\nabla}$ is nice and the matrix of J in a D^θ-flat frame $s^\theta = (s_i^\theta)$ defines a pluriharmonic map $\tilde{J}^\theta : (M, J, \bar{\nabla}) \to \mathcal{J}(V, \langle\cdot,\cdot\rangle) \to SO_0(2p, 2q)/U(p, q)$. Moreover, the map $\tilde{J}^\theta$ is harmonic.*

Proof. First we show, that $\overline{\nabla}$ is nice. Therefore we rewrite the Nijenhuis tensor

$$\begin{aligned}N_J(X, Y) &= (\nabla_{JX}J)Y - (\nabla_{JY}J)X - J(\nabla_X J)Y + J(\nabla_Y J)X\\ &= -4J(\nabla_X J)Y,\end{aligned}$$

where the second equality follows from the nearly Kähler condition and by

$$(\nabla_{JX}J)Y = -(\nabla_Y J)JX = J(\nabla_Y J)X = -J(\nabla_X J)Y.$$

But the torsion of $\overline{\nabla}$ is by Eq. (4.7)

$$T^{\overline{\nabla}}(X, Y) = -J(\nabla_X J)Y.$$

This shows that $\overline{\nabla}$ is nice.
By Corollary 3 the map $\tilde{J}^\theta$ is $\mathbb{S}^1$-pluriharmonic. Since $\overline{\nabla}$ is nice, Theorem 5 implies that $\tilde{J}^\theta$ is pluriharmonic. From the skew-symmetry of S and Proposition 7 we obtain that $\tilde{J}^\theta$ is harmonic. □

6.2. The dual Gauß map of a special Kähler manifold

In this subsection we consider a simply connected almost hermitian manifold (M, J, g) with a flat connection ∇, such that (∇, J) is special and the two-form $\omega = g(J\cdot, \cdot)$ is ∇-parallel.
Using the flat connection ∇ we identify by fixing a ∇-parallel symplectic frame s_0 the tangent space (TM, ω) with $(M \times V, \omega_0)$ where $V = \mathbb{R}^{2n}$ and ω_0 is its standard symplectic form.
The compatible complex structure J is seen as a map

$$J : M \to \mathfrak{J}(V, \omega_0),$$

where $\mathfrak{J}(V, \omega_0)$ is the set of complex structures on V which are compatible with ω_0.
Now we discuss the differential geometry of this set:
First we consider $\mathfrak{J}(V, \omega_0)$ as a subset of the vector space $\mathfrak{sp}(\mathbb{R}^{2n}) \subset \mathrm{Mat}(\mathbb{R}^{2n})$ characterized by the set of equations

$$f(j) = -\mathbb{1}_{2n}, \tag{6.4}$$

where $f : \mathrm{Mat}(\mathbb{R}^{2n}) \to \mathrm{Mat}(\mathbb{R}^{2n})$ is given by $f : A \mapsto A^2$. The differential of this map is $\mathrm{d}f_A(H) = \{A, H\}$ for $A, H \in \mathrm{Mat}(\mathbb{R}^{2n})$. In addition, $\mathrm{d}f$ has constant rank in points j satisfying Eq. (6.4), since one sees

$$\ker \mathrm{d}f_j = \{A \in \mathfrak{sp}(\mathbb{R}^{2n}) \mid \{j, A\} = 0\},$$
$$\mathrm{im}\, \mathrm{d}f_j \cong \{A \in \mathfrak{sp}(\mathbb{R}^{2n}) \mid [j, A] = 0\} \cong \mathfrak{u}(p, q).$$

Applying the regular value theorem we obtain that $\mathfrak{J}(V, \omega_0)$ is a submanifold of $\mathfrak{sp}(\mathbb{R}^{2n})$. Its tangent space at $j \in \mathfrak{J}(V, \omega_0)$ is

$$T_j\mathfrak{J}(V, \omega_0) = \ker \mathrm{d}f_j = \{A \in \mathfrak{sp}(\mathbb{R}^{2n}) \mid \{j, A\} = 0\}. \tag{6.5}$$

In addition the manifold $\mathfrak{J}(V, \omega_0)$ can be identified with the pseudo-Riemannian symmetric space $\mathrm{Sp}(\mathbb{R}^{2n})/U(p, q)$, where (p, q) is the hermitian signature of the hermitian metric $g(\cdot, \cdot) = \omega(J\cdot, \cdot)$, by the map

$$\Phi : \mathrm{Sp}(\mathbb{R}^{2n})/U(p, q) \to \mathfrak{J}(V, \omega_0),$$
$$gK \mapsto g j_0 g^{-1},$$

which maps the canonical base point $o = eK$ to j_0.
Any $j \in \mathfrak{J}(V, \omega_0)$ defines a symmetric decomposition of $\mathfrak{sp}(\mathbb{R}^{2n})$ by

$$\mathfrak{p}(j) = \{A \in \mathfrak{sp}(\mathbb{R}^{2n}) \mid \{j, A\} = 0\},$$
$$\mathfrak{k}(j) = \{A \in \mathfrak{sp}(\mathbb{R}^{2n}) \mid [j, A] = 0\} \cong \mathfrak{u}(p, q).$$

In particular $\mathfrak{k}(j_0) = \mathfrak{u}(p, q)$. Moreover, one observes $T_j\, \mathfrak{J}(V, \omega_0) = \mathfrak{p}(j)$.
Let $\tilde{j} \in \mathrm{Sp}(\mathbb{R}^{2n})/U(p, q)$ and $j = \Phi(\tilde{j})$, then $T_{\tilde{j}}\mathrm{Sp}(\mathbb{R}^{2n})/U(p, q)$ is canonically identified with $\mathfrak{p}(j)$ and for the differential of the identification one obtains

Proposition 9. *Let $\Psi = \Phi^{-1} : \mathfrak{J}(V, \omega_0) \to \mathrm{Sp}(\mathbb{R}^{2n})/U(p, q)$. Then it holds at $j \in \mathfrak{J}(V, \omega_0)$ that*

$$\mathrm{d}\Psi : T_j\, \mathfrak{J}(V, \omega_0) \ni X \mapsto -\frac{1}{2} j^{-1} X \in \mathfrak{p}(j). \tag{6.6}$$

This can be used to relate the differential of a map

$$J : M \to \mathfrak{J}(V, \omega_0)$$

and a map

$$\tilde{J} = \Psi \circ J : M \to \mathrm{Sp}(\mathbb{R}^{2n})/U(p, q)$$

by

$$\mathrm{d}\tilde{J} = -\frac{1}{2} J^{-1} \mathrm{d}J.$$

Recall, that under the above assumptions $(TM, D = \nabla - S, S = \frac{1}{2}J(\nabla J), g)$ defines a metric tt^*-bundle. Analogous to the last section we obtain

Theorem 8. *Let (M, J, g) be a simply connected almost hermitian manifold with a flat connection ∇, such that (∇, J) is special and the two-form $\omega = g(J\cdot, \cdot)$ is ∇-parallel and let $(TM, D = \nabla - S, S = \frac{1}{2}J(\nabla J), g)$ be the associated metric tt^*-bundle. Then the matrix of J in a D^θ-flat frame $s^\theta = (s_i^\theta)$ defines an $\mathbb{S}^1$-pluriharmonic map $\tilde{J}^\theta : M \to \mathcal{J}(V, \omega_0) \to \mathrm{Sp}(\mathbb{R}^{2n})/U(p, q)$.*
In particular, given a nice connection D on (M, J) then the map $\tilde{J}^\theta : (M, J, D) \to \mathrm{Sp}(\mathbb{R}^{2n})/U(p, q)$ is pluriharmonic.

Proof. Since $D^0\omega = \nabla\omega = (D + S)\omega = 0$ and $S_X^\theta := \cos(\theta)S_X + \sin(\theta)S_{JX}$ is skew-symmetric with respect to ω, we obtain $D\omega = 0$ and $D^\theta\omega = 0$. Therefore we can choose for each θ the D^θ-parallel frame s^θ as a symplectic frame, such that $s^{\theta=0} = s_0$. This yields using $DJ = 0$

$$X.\omega(Js_i^\theta, s_j^\theta) = \omega(D_X^\theta(Js_i^\theta), s_j^\theta) = \omega((D_X^\theta J)s_i^\theta, s_j^\theta) = \omega([S_X^\theta, J]s_i^\theta, s_j^\theta) = -2\omega(JS_X^\theta s_i^\theta, s_j^\theta).$$

Let S^{s^θ}, J^{s^θ} be the representation of S and J in the frame s^θ, then

$$(J^{s^\theta})^{-1}X(J^{s^\theta}) = -2S^{s^\theta}$$

or

$$\mathrm{d}\tilde{J}^\theta = (s^\theta)^{-1} \circ S^\theta \circ s^\theta,$$

where the frame s^θ is seen as a map $s^\theta : M \times V \to TM$. This shows for $X \in \Gamma(TM)$

$$\begin{aligned}\mathrm{d}\tilde{J}^\theta(X) &= (s^\theta)^{-1} \circ S_X^\theta \circ (s^\theta) = (s^\theta)^{-1} \circ S_{\mathcal{R}_\theta X} \circ (s^\theta)\\ &= ((s^\theta)^{-1}s^0) \circ \mathrm{d}\tilde{J}(\mathcal{R}_\theta X) \circ ((s^0)^{-1}s^\theta)\\ &= Ad_{\alpha_\theta}^{-1} \circ \mathrm{d}\tilde{J}(\mathcal{R}_\theta X) = \Phi_\theta^{-1} \circ \mathrm{d}\tilde{J}(\mathcal{R}_\theta X),\end{aligned}$$

where $\alpha_\theta = (s^\theta)^{-1}s^0$ is the frame change from s_0 to s_θ and $\Phi_\theta = Ad_{\alpha_\theta}$ which is parallel with respect to the Levi-Civita connection on $\mathrm{Sp}(\mathbb{R}^{2n})/U(p, q)$. In other words we have found an associated family. Given a nice connection D on (M, J) Theorem 5 shows that $\tilde{J}^\theta$ is pluriharmonic. □

If the above tt^*-bundle comes from a special Kähler manifold we have the

Theorem 9. *Let (M, J, g, ∇) be a special Kähler manifold and $(TM, D = \nabla - S, S = \frac{1}{2}J(\nabla J), g)$ the associated metric tt^*-bundle, then the matrix of J in a D^θ-flat frame $s^\theta = (s_i^\theta)$ defines a pluriharmonic map $\tilde{J}^\theta : (M, J, D) \to \mathrm{Sp}(\mathbb{R}^{2n})/U(p, q)$. Moreover, $\tilde{J}^\theta$ is harmonic.*

Proof. By Theorem 8 the map $\tilde{J}^\theta$ is $\mathbb{S}^1$-pluriharmonic. In the special Kähler case we know that D is the Levi-Civita connection and hence torsion-free. The complex structure J is integrable and so $N_J = 0$. This means, that D is nice and Theorem 5 shows that $\tilde{J}^\theta$ is pluriharmonic. Since S is trace-free we get from Proposition 7 that $\tilde{J}^\theta$ is harmonic. □

In [4] we studied this pluriharmonic/harmonic map for a special Kähler manifold in more details.

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References

[1] D.V. Alekseevsky, V. Cortés, C. Devchand, Special complex manifolds, J. Geom. Phys. 42 (2002) 85–105.
[2] N. Abe, S. Kurosu, A decomposition of a holomorphic vector-bundle with connection and its applications to complex affine immersions, Results Math. 44 (2003) 3–24.

[3] V. Cortés, Specical Kähler manifolds: A survey, in: Proc. of the 21st Winter School 'Geometry and Physics', Srni (2001), Rendi. Circ. Mat. d Palermo. Serie II 69 (2002).
[4] V. Cortés, L. Schäfer, Topological-Antitopological Fusion Equations, Pluriharmonic Maps and Special Kähler Manifolds, in: Progress in Mathematics, vol. 234, Birkhäuser, 2005.
[5] V. Cortés, L. Schäfer, Flat Nearly Kähler Manifolds (in press).
[6] B. Dubrovin, Geometry and integrability of topological-antitopological fusion, Comm. Math. Phys. 152 (1993) 539–564.
[7] B. de Wit, A. Van Proeyen, Potentials and symmetries of general guaged $N = 2$ supergravity-Yang-Mills models, Nuclear Phys. B 245 (1984) 89–117.
[8] J.-H. Eschenburg, R. Tibuzy, Associated families of pluriharmonic maps and isotropy, Manuscripta Math. 95 (3) (1998) 295–310.
[9] D.S. Freed, Special Kähler manifolds, Comm. Math. Phys. 203 (1) (1999) 31–52.
[10] Th. Friedrich, S. Ivanov, Parallel spinors and connections with skew-symmetric torsion in string theory, Asian J. Math. 6 (2) (2002) 303–335.
[11] S. Kobayashi, K. Nomizu, Foundations of Differential Geometry Volume I/II, Interscience Publishers, 1963–1969.
[12] M. Krahe, Ph.D. Thesis (in preparation).
[13] M.-A. Lawn, L. Schäfer, Decompositions of para-complex vector-bundles and affine para-complex immersions, Res. in Math. 48 (3–4) (in press).
[14] D. Mc Duff, D. Salamon, Introduction to Symplectic Topology, Oxford University Press, 1995.
[15] P.-A. Nagy, On nearly-Kähler geometry, Ann. Global Anal. Geom. 22 (2) (2002) 167–178.
[16] P.-A. Nagy, Nearly Kähler geometry and Riemannian foliations, Asian J. Math. 6 (3) (2002) 481–504.
[17] L. Schäfer, tt^*-bundles and pluriharmonic maps, Ann. Global Anal. Geom. 28 (3) (2005).
[18] L. Schäfer, Harmonic bundles, topological-antitopological fusion and the related pluriharmonic maps, J. Geom. Phys. 56 (5) (2006) 830–842.